Sample Exam Style

Solutions

1. Evaluate $\int_1^5 x^2 e^{2x}\,dx$.

Solution: $\displaystyle\int_1^5 x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x}\Big|_1^5 - \int_1^5 e^{2x} x\,dx$

$$u_1 = x^2 \qquad v_1 = \frac{1}{2}e^{2x}$$
$$du_1 = 2x\,dx \qquad dv_1 = e^{2x}\,dx$$

$$u_2 = x \qquad v_2 = \frac{1}{2}e^{2x}$$
$$du_2 = dx \qquad dv_2 = e^{2x}\,dx$$

$$= \frac{1}{2}x^2 e^{2x}\Big|_1^5 - \left(\frac{1}{2}xe^{2x}\Big|_1^5 - \int_1^5 e^{2x}\,dx\right)$$

$$= \frac{1}{2}x^2 e^{2x}\Big|_1^5 - \left(\frac{1}{2}xe^{2x}\Big|_1^5 - \frac{1}{2}e^{2x}\Big|_1^5\right)$$

$$= \left(\frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}e^{2x}\right)\Big|_1^5$$

$$= \left(\frac{1}{2}\cdot 5^2 e^{10} - \frac{1}{2}\cdot 5\cdot e^{10} + \frac{1}{2}e^{10}\right) - \left(\frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{2}e^2\right)$$

$$= \left(\frac{25}{2} - \frac{5}{2} + \frac{1}{2}\right)e^{10} - \frac{1}{2}e^2$$

$$= \boxed{\frac{21}{2}e^{10} - \frac{1}{2}e^2}$$

Any stage here is sufficient

2. Evaluate $\int \sin(x)e^x\,dx$

Solution:

$f_1(x) = \sin(x)$	$g_1(x) = e^x$
$f_1'(x) = \cos(x)$	$g_1'(x) = e^x$
$f_2(x) = \cos(x)$	$g_2(x) = e^x$
$f_2'(x) = -\sin(x)$	$g_2'(x) = e^x$

$$
\begin{aligned}
I &= \int \sin(x)e^x\,dx \\
&= \sin(x)e^x - \int \cos(x)e^x\,dx \\
&= \sin(x)e^x - \left(\cos(x)e^x + \int \sin(x)e^x\,dx\right) \\
&= \sin(x)e^x - \cos(x)e^x - I
\end{aligned}
$$

$$2I = \sin(x)e^x - \cos(x)e^x + C$$

$$\boxed{I = \underbrace{\frac{1}{2}e^x\left(\sin(x) - \cos(x)\right)} + C}$$

This piece doesn't have to be factored in this particular way.

3. Evaluate $\int (\tan(x))^3\,dx$.

Solution:

$$\int (\tan(x))^3\,dx = \int \tan(x)\left((\sec(x))^2 - 1\right)dx$$

$u = \tan(x)$

$du = (\sec(x))^2\,dx$

$$= \int \tan(x)(\sec(x))^2\,dx - \int \tan(x)\,dx$$

$$= \int u\,du - \int \tan(x)\,dx$$

$$= \frac{u^2}{2} - \int \tan(x)\,dx$$

$$= \frac{(\tan(x))^2}{2} - \int \tan(x)\,dx$$

$$= \boxed{\frac{(\tan(x))^2}{2} - \underbrace{\ln|\sec(x)|}_{} + C.}$$

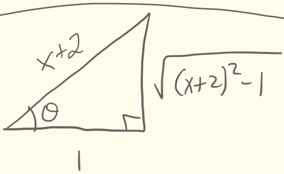
$-\ln|\sec(x)| = \ln|\cos(x)|$

either is acceptable.

4. Evaluate $\int \sqrt{x^2+4x+3}\, dx$.

Solution: $x^2+4x+3 = (x+2)^2 - 1$

$$\int \sqrt{x^2+4x+3}\, dx = \int \sqrt{(x+2)^2-1}\, dx$$



$\sec(\theta) = x+2$

$dx = \sec\theta \tan(\theta)\, d\theta$

$$= \int \sqrt{(\sec\theta)^2-1}\ \sec\theta \tan\theta\, d\theta$$

$$= \int (\tan(\theta))^2 \sec\theta\ d\theta$$

$$= \int ((\sec\theta)^2 - 1)\sec\theta\ d\theta$$

$$= \int (\sec\theta)^3\, d\theta - \int \sec\theta\, d\theta$$

$u_1 = \sec\theta \qquad v_1 = \tan\theta$

$du_1 = \sec\theta\tan\theta\, d\theta \qquad dv = (\sec\theta)^2 d\theta$

$$= -\ln|\sec\theta+\tan\theta| + \int (\sec\theta)^3 d\theta$$

$$= -\ln|\sec\theta+\tan\theta| + \sec\theta\tan\theta - \int (\tan\theta)^2 \sec\theta\, d\theta$$

$$2\int (\tan\theta)^2 \sec\theta\, d\theta = \sec\theta\tan\theta - \ln|\sec\theta+\tan\theta| + C$$

$$\int (\tan\theta)^2 \sec\theta\, d\theta = \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right) + C$$

$$\| $$

$$\int \sqrt{x^2+4x+3}\, dx = \boxed{\frac{1}{2}\left((x+2)\sqrt{(x+2)^2-1} - \ln\left|x+2+\sqrt{(x+2)^2-1}\right|\right) + C}$$

Make sure that you tie what you were asked to compute to what you want your answer to be since we broke the chain of equalities.

5. Evaluate $\int \frac{1}{x^3+2x^2+x}\,dx$.

Solution:

$$\frac{1}{x^3+2x^2+x} = \frac{1}{x(x^2+2x+1)} = \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$

$$1 = A(x^2+2x+1) + Bx^2 + Bx + Cx$$

$$1 = Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$$

x^2: $0 = A + B$

x: $0 = 2A + B + C$

c: $1 = A$

$0 = 1 + B$

$B = -1$

$0 = 2\cdot 1 - 1 + C$

$0 = 1 + C$

$C = -1$

$$\int \frac{1}{x^3+2x^2+x}\,dx = \int \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$$

$$= \boxed{\ln|x| \underbrace{- \ln|x+1| + \frac{1}{x+1}} + C}$$

If these two substitutions are not obvious, you should write them out.

6. Determine if $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ converges or diverges. If the improper integral converges, determine its value.

Solution:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

$$\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to-\infty} \left(\arctan(x) \Big|_{t}^{0} \right)$$

$$= \lim_{t\to-\infty} \left(\arctan(0) - \arctan(t)\right)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \arctan(x) \Big|_{0}^{t}$$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(0)$$

$$= \frac{\pi}{2} - 0$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

Converges

7. Determine if $\int_1^\infty \frac{1+3(\sin(2x))^4}{\sqrt{x}}\,dx$ converges or diverges.

Solution: Compare with $\frac{1}{\sqrt{x}}$] You don't have to write this but I like to.

$$-1 \le (\sin(2x)) \le 1$$

$$0 \le (\sin(2x))^4 \le 1$$

$$0 \le 3(\sin(2x))^4 \boxed{\le 3}$$

$$1 \le 1+3(\sin(2x))^4 \boxed{\le 4}$$

I'm not actually going to use these upper bounds.

Hypotheses

$$0 \le \frac{1}{\sqrt{x}} \le \frac{1+3(\sin(2x))^4}{\sqrt{x}}$$

$\frac{1}{\sqrt{x}}$, $\frac{1+3(\sin(2x))^4}{\sqrt{x}}$ continuous on $[1,\infty)$

$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges (p-test, $p=\frac{1}{2}\le 1$)

By comparison test, $\int_1^\infty \frac{1+3(\sin(2x))^4}{\sqrt{x}}\,dx$ also diverges.

Name of test

result of test